

Spin and orbital angular momentum in gauge theories (I): QED and determination of the angular momentum density

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This two-paper series addresses and fixes the long-standing gauge invariance problem of angular momentum in gauge theories. This QED part reveals: 1) The spin and orbital angular momenta of electrons and photons can all be consistently defined gauge invariantly. 2) These gauge-invariant quantities can be conveniently computed via the canonical, gauge-dependent operators (e.g., $\psi^\dagger \vec{x} \times \frac{1}{i} \vec{\nabla} \psi$) in the Coulomb gauge, which is in fact what people (unconsciously) do in atomic physics. 3) The renowned formula $\vec{x} \times (\vec{E} \times \vec{B})$ is a wrong density for the electromagnetic angular momentum. The angular distribution of angular-momentum flow in polarized atomic radiation is properly described not by this formula, but by the gauge invariant quantities defined here. The QCD paper [1] will give a non-trivial generalization to non-Abelian gauge theories, and discuss the connection to nucleon spin structure.

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Energy, momentum, and angular momentum are among the most important properties of a physical field, in both classical and quantum theories. It should be emphasized that these quantities are not uniquely determined by the field Lagrangian. Their definitions allow for certain arbitrariness, which must be fixed by other physical requirements or by experiments. Gauge invariance is one such requirement. It helps to fix the form of the energy-momentum tensor, which is to be coupled to the gravitational field, therefore must have a gauge invariant *density*. Certainly the gauge invariance criteria also applies to the angular momentum. But up to now it seemed to bring more uneasy feelings rather than assistance. For one example, the labeling of atomic states employs the electron orbital angular momentum operator $\vec{L}_e = \int d^3x \psi^\dagger \vec{x} \times \frac{1}{i} \vec{\nabla} \psi$, but this operator is gauge dependent! For another example, it was taught in common textbooks that gauge invariance prohibits the separation of photon angular momentum into spin and orbital contributions. [2, 3]

The goal of this paper is to turn the situation around. Our strategy is to investigate the spin and orbital angular momenta of electrons and photons in a whole QED system, and examine the angular momentum at the same footing as the energy-momentum, namely, by seeking an appropriate density expression. We will see that examination of the standard angular distribution in polarized atomic radiation can actually do the job.

We start with the “canonical” angular momentum op-

erators in QED:

$$\begin{aligned} \vec{J}_{QED} &= \int d^3x \psi^\dagger \frac{1}{2} \vec{\Sigma} \psi + \int d^3x \psi^\dagger \vec{x} \times \frac{1}{i} \vec{\nabla} \psi \\ &+ \int d^3x \vec{E} \times \vec{A} + \int d^3x E^i \vec{x} \times \vec{\nabla} A^i \\ &\equiv \vec{S}_e + \vec{L}_e + \vec{S}_\gamma + \vec{L}_\gamma. \end{aligned} \quad (1)$$

These operators are termed “canonical” ones because their individual roles in generating spatial rotations can be directly recognized. These expressions are derived straightforwardly by applying Nöther’s theorem to the QED Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i\gamma^\mu D_\mu - m) \psi. \quad (2)$$

The canonical angular momentum operators in Eq. (1) are what people use familiarly in discussing polarized atomic states and radiations. However, except for the electron spin, all other three terms are gauge dependent. This brings a very uneasy concern about the physical meanings of these quantities, and even the whole labeling of atomic states.

Before the above subtlety was clearly up, its counterpart in QCD was sharply faced when people tried to understand the nucleon spin in terms of the spin and orbital contributions of quarks and gluons. [4] To satisfy the gauge invariance requirement, people considered an alternative, explicitly gauge invariant decomposition of the QCD angular momentum. [5, 6] Its QED version is

$$\vec{J}_{QED} = \int d^3x \psi^\dagger \frac{1}{2} \vec{\Sigma} \psi + \int d^3x \psi^\dagger \vec{x} \times \frac{1}{i} \vec{D} \psi$$

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$$\begin{aligned}
& + \int d^3x \vec{x} \times (\vec{E} \times \vec{B}) \\
& \equiv \vec{S}_e + \vec{L}'_e + \vec{J}'_\gamma.
\end{aligned} \tag{3}$$

This is obtained from Eq. (1) by adding a surface term which vanishes after integration. At first sight, this decomposition might appear satisfactory, because $\vec{x} \times (\vec{E} \times \vec{B})$ is the familiar electromagnetic angular momentum constructed with the renowned Poynting vector. However, the gauge invariance of Eq. (3) does not help to justify the labeling of atomic states, which uses eigenvalues of the canonical \vec{L}'_e , not the gauge invariant \vec{L}'_e . It does not answer either the physical meanings of the photon spin and orbital angular momentum, especially the measurements for them: The photon spin has been measured directly by Beth over 70 years ago. [9] Recently, detection and manipulation of the photon orbital angular momentum have also been carried out, and became a hot topic due to its potential application in quantum information processing. [10, 11, 12, 13, 14, 15]

As we advocated at the beginning, we seek to remove the arbitrariness in defining angular momentum by examining the density. When brought to this trial, Eqs. (1) and (3) cannot possibly both survive, because they give different angular momentum densities, although the integrated \vec{J}_{QED} is the same. Unlike the energy-momentum tensor, the rank-3 angular momentum tensor does not couple to any physical field, so no theoretical clue can be found in this regard. We have to ask what kind of experiments can measure the angular momentum density. The answer we provide here is the standard angular distribution in polarized atomic radiations.

Let us consider a pure electric multipole radiation of order (l, m) . (The discussion for magnetic-type radiation is exactly similar.) The photon wave-function is given by

$$\begin{aligned}
\vec{B}_{lm} &= a_{lm} j_l(kr) \vec{L} Y_{lm} \\
\vec{E}_{lm} &= ik \vec{A}_{lm} = \frac{i}{k} \vec{\nabla} \times \vec{B}_{lm}.
\end{aligned} \tag{4}$$

Here \vec{A}_{lm} is written in the Coulomb gauge. $\vec{L} \equiv \vec{x} \times \frac{1}{i} \vec{\nabla}$. The wave amplitude a_{lm} is related to the emission probability, which in turn is determined by the transition matrix element of the electric 2^l -pole moment. The angular distribution of this radiation is

$$\frac{dP}{d\Omega} = \frac{1}{2k^2} \cdot \frac{1}{l(l+1)} |a_{lm}|^2 |\vec{L} Y_{lm}|^2. \tag{5}$$

The flow of angular momentum must also follow this distribution, because an emitted photon takes off all the energy and angular momentum lost by the polarized source atom. Now we are at the position to check which definition of angular momentum density can produce the above angular distribution.

Before promptly discarding the gauge-dependent density given in Eq. (1), a moment's thought tells that it can actually agree perfectly with Eq. (5), but only in the

Coulomb gauge, in which the photon wave-function \vec{A}_{lm} in Eq. (4) is the eigenstate of $(J_\gamma)_z = (S_\gamma)_z + (L_\gamma)_z$.

Then, how about the gauge invariant $\vec{x} \times (\vec{E} \times \vec{B})$. Awkwardly, this renowned formula performs badly. Take the simplest electric dipole radiation as an example. Setting $(l, m) = (1, 1)$ in Eq. (5) gives

$$\frac{dP}{d\Omega} = k \frac{dJ_z}{d\Omega} = \frac{1}{2k^2} \cdot \frac{3}{8\pi} |a_{lm}|^2 \cdot \frac{1}{2} (1 + \cos^2 \theta). \tag{6}$$

However, $\vec{x} \times (\vec{E} \times \vec{B})$ leads to a flow of angular momentum projection J_z according to

$$\frac{dJ_z}{d\Omega} = \frac{1}{2k^3} \cdot \frac{3}{8\pi} |a_{lm}|^2 \sin^2 \theta. \tag{7}$$

This strongly disagree with Eq. (6), and is evidently wrong: In electric $(1, 1)$ radiation each emitted photon takes off $1\hbar$ of J_z , but Eq. (7) says that no J_z flows along the z axis, whereas along the x - y plane one emitted photon should carry $2\hbar$ of J_z .

In fact, there had been various clear hints that $\vec{x} \times (\vec{E} \times \vec{B})$ is not a correct description of the electromagnetic angular momentum density. (The most well-known example is probably the plane-wave paradox.) In [6], two of us (Chen and Wang) pointed out that when the electromagnetic field interacts with the Dirac field, $J'_\gamma = \int d^3x \vec{x} \times (\vec{E} \times \vec{B})$ is not its rotation generator.

This can be easily seen from Eq. (3): The total \vec{J}_{QED} and the electron spin $\int d^3x \psi^\dagger \frac{1}{2} \vec{\Sigma} \psi$ both satisfy the angular momentum algebra $\vec{J} \times \vec{J} = i\vec{J}$, but since this algebra is obviously violated by the operator $\vec{L}'_e = \int d^3x \psi^\dagger \vec{x} \times \frac{1}{i} \vec{\nabla} \psi$, it must also be violated by the remaining term J'_γ , which therefore cannot possibly be a rotation generator.

Now that we have ruled out the explicitly gauge-invariant Eq. (3), while Eq. (1) can only do a good job in a specific (the Coulomb) gauge, where is the satisfactory solution? To a large extent, the solution had actually been found over ten year ago. [7] (See also a recent discussion in [8].) When organized consistently, the gauge invariant, physically reasonable expression for QED angular momentum reads

$$\begin{aligned}
\vec{J}_{QED} &= \int d^3x \psi^\dagger \frac{1}{2} \vec{\Sigma} \psi + \int d^3x \psi^\dagger \vec{x} \times \frac{1}{i} \vec{\nabla} \psi \\
&+ \int d^3x \vec{E} \times \vec{A}_{phys} + \int d^3x E^i \vec{x} \times \vec{\nabla} A^i_{phys} \\
&\equiv \vec{S}_e + \vec{L}''_e + \vec{S}''_\gamma + \vec{L}''_\gamma.
\end{aligned} \tag{8}$$

Here, $\vec{D}_{pure} \equiv \vec{\nabla} - ie\vec{A}_{pure}$, $\vec{A}_{pure} + \vec{A}_{phys} \equiv \vec{A}$ are defined through:

$$\vec{\nabla} \cdot \vec{A}_{phys} = 0, \tag{9}$$

$$\vec{\nabla} \times \vec{A}_{pure} = 0. \tag{10}$$

They are nothing but the transverse and longitudinal components of the vector potential \vec{A} . The suffixes we use

here is to make their physical contents clear, and to make preparation for generalizations to QCD [1]. With the boundary condition that \vec{A} , \vec{A}_{pure} , and \vec{A}_{phys} all vanish at spatial infinity, Eqs. (9) and (10) prescribe a unique decomposition of \vec{A} into \vec{A}_{pure} and \vec{A}_{phys} , and dictates their gauge transformation properties:

$$\vec{A}_{pure} \rightarrow \vec{A}'_{pure} = \vec{A}_{pure} + \vec{\nabla}\Lambda, \quad (11)$$

$$\vec{A}_{phys} \rightarrow \vec{A}'_{phys} = \vec{A}_{phys}. \quad (12)$$

Eqs. (10) and (11) tell that in QED \vec{A}_{pure} is a pure gauge field in all gauges, and it transforms in the same manner as the full vector field \vec{A} does:

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\Lambda. \quad (13)$$

On the other hand, the transverse field \vec{A}_{phys} is unaffected by gauge transformation, thus can be regarded as the “physical” part of \vec{A} .

Now we have all the need elements to explain that Eq. (8) gives the correct expressions of spin and orbital angular momenta of electrons and photons, including their densities. First of all, the total \vec{J}_{QED} given by Eq. (8) equals that in Eqs. (1) and (3). This can be proved by writing $\vec{B} = \vec{\nabla} \times \vec{A}_{phys}$ in Eq. (3), performing an integration by parts, and rearranging the results. Secondly, the gauge transformation properties of \vec{A}_{pure} and \vec{A}_{phys} tell that each *density* term in Eq. (8) is separately gauge invariant (certainly so is the integrated operator). Thirdly, like the canonical \vec{L}_e , the gauge invariant \vec{L}_e'' satisfies the algebra $\vec{J} \times \vec{J} = i\vec{J}$. This is due to the property of \vec{A}_{pure}

in Eq. (10). And finally, we note that in the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$, the longitudinal (pure-gauge) field \vec{A}_{pure} vanishes, thus all quantities in Eq. (8) coincide with their canonical counterparts in Eq. (1). This observation is of vital importance. It reveals that the gauge invariant quantities in Eq. (8) can all be conveniently computed via the canonical operators in the Coulomb gauge. This is actually what people (unconsciously) do in studying atomic and electromagnetic angular momenta, including the recent measurements of the photon orbital angular momentum [10, 11, 12, 13, 14, 15]. It is thus understandable why these studies always get reasonable results.

After confirming that Eq. (8) is indeed the correct and satisfactory answer for angular momenta in QED, one may feel like to talk some “latter-wit” about it: The form of Eq. (8) could have been guessed out by reasonable physical considerations: The photon angular momentum should contain only the “physical” part of the gauge field, which nevertheless should not appear in the orbital angular momentum of the electron. The latter should thus only include the non-physical \vec{A}_{pure} to cancel the also non-physical phase dependence of the electron field, keeping the whole \vec{L}_e'' gauge invariant. Honestly, these “physical considerations” hardly helped us in writing down Eq. (8), but such “latter-wit” for QED did serve as an important guidance to a non-trivial solution for the angular momentum in QCD [1].

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- [1] X.S. Chen, X.F. Lü, W.M. Sun, F. Wang, and T. Goldman, *Spin and orbital angular momentum in gauge theories (II): QCD and nucleon spin structure*, arXiv: 0709.1284 [hep-ph].
 - [2] J.M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, 1955).
 - [3] V.B. Berestetskii, E.M. Lifshitz, and L.P. Pitaevskii, *Quantum Electrodynamics*, 2nd ed. (Butterworth-Heinemann, 1982).
 - [4] The European Muon Collaboration (J. Ashman *et al*), Phys. Lett. **B206**, 364 (1988); Nucl. Phys. **B328**, 1 (1989).
 - [5] X. Ji, Phys. Rev. Lett. **78**, 610 (1997).
 - [6] X.S. Chen and F. Wang, Commun. Theor. Phys. **27**, 121 (1997).
 - [7] S.J. van Enk and G. Nienhuis, Europhys. Lett. **25**, 497 (1994); J. Mod. Opt. **41**, 963 (1994).
 - [8] G.F. Calvo, A. Picón, and E. Bagan, Phys. Rev. A **73**, 013805 (2006).
 - [9] R.A. Beth, Phys. Rev. **50**, 115 (1936).
 - [10] S.J. van Enk and G. Nienhuis, arXiv: 0707.3434 [quant-ph].
 - [11] M.F. Andersen *et al*, Phys. Rev. Lett., **97**, 170406 (2006).
 - [12] A. Alexandrescu, D. Cojoc, and E. Di Fabrizio, Phys. Rev. Lett. **96**, 243001 (2006).
 - [13] L. Marrucci, C. Manzo, and D. Paparo, Phys. Rev. Lett. **96**, 163905 (2006).
 - [14] J. Leach, M.J. Padgett, S.M. Barnett, S. Franke-Arnold, and J. Courtial, Phys. Rev. Lett. **88**, 257901, (2002).
 - [15] L. Allen, M.W. Beijersbergen, R.J.C. Spreeuw, and J.P. Woerdman, Phys. Rev. A **45**, 8185 (1992).